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1998 J. Phys. A: Math. Gen. 31 5565

(http://iopscience.iop.org/0305-4470/31/25/007)

J. Phys. A: Math. Gen. **31** (1998) 5565–5575. Printed in the UK PII: S0305-4470(98)90763-6

Finite density effect in the Gross–Neveu model on weakly curved surfaces

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Received 16 January 1998

Abstract. The three-dimensional Gross–Neveu model in $R^1 \times M^2$ spacetime, where M^2 is a weakly curved two-dimensional surface, is investigated, using an effective potential at a finite curvature R and nonzero chemical potential μ. The critical values of (R, μ) are derived, such that a system undergoes the first-order phase transition from the phase with broken chiral invariance to the symmetric phase. The fermion density is found to be of nonanalytic behaviour at the critical value of the chemical potential.

1. Introduction

Recently four-fermion field theories in $(2 + 1)$-dimensional Minkowski spacetime, which are known as Gross–Neveu (GN) models [1], are under extensive investigation for purely theoretical motivation and also because of their applications to planar condensed matter physics. Such theories possess many desirable properties: the renormalizability in the $1/N$ expansion, dynamical breaking of chiral symmetry and generation of fermion mass for a large coupling constant as in QCD [2], the analogy to the BCS theory of superconductivity in two spatial dimensions and the possibility of describing a new phenomenon of high-temperature superconductivity [3], the reduction to the $S = \frac{1}{2}$ quantum antiferromagnet Heisenberg model in the continuum limit [4] and so on. The main features of these models, obtained using the large N expansion technique, are confirmed within the framework of other nonperturbative approaches [5].

Since there are no closed physical systems in nature, the influence of different external factors on the vacuum of the simplest GN model was considered. In [6] some critical phenomena of this theory were studied at nonzero temperature T and chemical potential μ.

Recently, on the same foundation a new property of external (chromo-)magnetic field H to promote the dynamical chiral symmetry breaking has been discovered [7]. (At present it is the well known effect of a dynamical chiral symmetry breaking catalyst by external magnetic field [8], which is under intensive consideration [9].) The role of T, H as well as of μ, H in the formation of a ground state of the GN model has also been clarified [10].

The study of dynamical symmetry breaking in spacetimes with curvature and nontrivial topology is also of great importance, since in the early universe the gravity was sufficiently strong and one should take it into account. A copious amount of literature on this subject is

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0305-4470/98/255565+11$19.50

available (see the review [11]). The effect of curvature and nontrivial topology on the chiral symmetry breaking in four-fermion models was first discussed in [12, 13]. The curvature-induced first-order phase transition from a chirally symmetric to a chirally nonsymmetric phase was shown to exist in those models in the linear curvature approximation. It turns out that in specific spacetimes such as the Einstein universe [14] and maximally symmetric spacetimes [15] the above-mentioned models can be solved exactly in the leading order of the large N expansion technique. Finally, dynamical symmetry breaking in the external gravitational and magnetic fields is considered in [16].

It is well known that low-dimensional four-fermion field theories, especially the $(2+1)$-dimensional GN model, in curved spacetimes [13, 17–19] and in the nonsimply connected spacetimes [20, 21] may be very useful for the investigation of physical processes in thin films and in the materials with layer structure. The point is that an external stress, applied to the planar system, can change the topology and curvature of a surface. A great amount of observable physical phenomena are due to nonzero particle density (superconductivity, quantum Hall effect, etc). Thus, here the influence of both chemical potential and curvature of space on the phase structure of the $(2+1)$-dimensional GN model is studied. In particular, we shall consider $R^1 \times M^2$ spacetime to clarify our discussion. Here M^2 is an arbitrary weakly curved noncompact two-dimensional spatial surface.

In section 2 we evaluate the one-loop effective potential in $R^1 \times M^2$ spacetime at nonzero chemical potential. In this we suppose that the surface M^2 curves slowly, so we only keep terms independent of curvature R and terms linear in R. Section 3 gives a detailed analysis of the effective potential, which shows the existence of a phase transition restoring the chiral symmetry of the system while the curvature R and chemical potential μ are varied. Finally, we summarize our results in section 4.

2. Effective potential in $R^1 \times M^2$ spacetime at $\mu \neq 0$

The four-fermion model in the $R^1 \times M^2$ spacetime, where M^2 is the two-dimensional weakly curved space, is described by the action [11, 13, 18]

$$S = \int \mathrm{d}^3x \sqrt{-g} \left[\mathrm{i}\bar{\psi}_j \gamma^\mu(x) \nabla_\mu \psi_j + \frac{\lambda^2}{2N} (\bar{\psi}_j \psi_j)^2 \right] \tag{1}$$

where g is the determinant of the spacetime metric $g_{\mu\nu}$, ∇_μ is the covariant derivative and the summation over j is implied ($j = 1, 2, \ldots, N$). Here fermion fields ψ_j are taken in the reducible four-dimensional representation of $SL(2, C)$. In this case the algebra of the γ-matrices is presented in [2]. This action has the discrete chiral symmetry,

$$\psi \to \gamma_5 \psi. \tag{2}$$

As a result, the chiral symmetry is maintained at any order of ordinary perturbation theory. However, as is evident from different nonperturbative approaches [1, 2, 5] the symmetry may be broken dynamically for large values of the coupling constant λ. To see the nonperturbative features such as spontaneous symmetry breaking and dynamical mass generation in this model, it is convenient to rewrite the above action in an equivalent form [1] by introducing the auxiliary field $\sigma(x)$,

$$S = \int \mathrm{d}^3x \sqrt{-g} \left[\mathrm{i}\bar{\psi}_j \gamma^\mu(x) \nabla_\mu \psi_j - \sigma \bar{\psi}_j \psi_j - \frac{N}{2\lambda^2} \sigma^2 \right]. \tag{3}$$

This expression suggests explicitly that the vacuum expectation value of the σ field plays the role of mass for the fermions. In order to find the effective potential in the theory

with the action of equation (1), we follow [12, 13] where this quantity was considered in a weak curvature approximation. First, let us integrate over the fermion fields in equation (3) and evaluate an effective action $S_{\rm eff}(\sigma)$ describing the self-interaction of the σ field:

$$\exp(\mathrm{i}NS_{\rm eff}(\sigma)) = \int \mathrm{D}\psi\,\mathrm{D}\bar{\psi}\,\exp[\mathrm{i}S(\psi,\bar{\psi},\sigma)]. \tag{4}$$

Here we use the $1/N$ expansion which is the fermion-loop expansion. In the mean-field approximation, where the $\sigma(x)$ field is assumed to be constant, and to the leading order in the large N, one can obtain the one-loop effective potential $U(\sigma)$ from the action $S_{\rm eff}(\sigma)$:

$$U(\sigma) = \frac{\sigma^2}{2\lambda^2} + \mathrm{i}\,\mathrm{tr}\langle x|\ln(\mathrm{i}\gamma^\mu(x)\nabla_\mu - \sigma)|x\rangle \tag{5}$$

where tr is over indices other than spacetime indices. Using the Green function $G_F(x,y;\sigma)$ defined by the relation

$$G_F(x,y;\sigma) \equiv \langle x|(\mathrm{i}\gamma^\mu\nabla_\mu - \sigma)^{-1}|y\rangle \tag{6}$$

we rewrite equation (5) as follows

$$U(\sigma) = \frac{\sigma^2}{2\lambda^2} - \mathrm{i}\,\mathrm{tr}\ln G_F(x,x;\sigma). \tag{7}$$

The logarithm can be eliminated from this equation by introducing the parameter s:

$$\ln\left[\frac{K-\sigma}{K}\right] = -\int_0^\sigma \mathrm{d}s\,\frac{1}{K-s} \tag{8}$$

where an operator K is given as $\mathrm{i}\gamma^\mu(x)\nabla_\mu$ in this case. Therefore, equation (7) is rewritten in the following form

$$U(\sigma) = \frac{\sigma^2}{2\lambda^2} - \mathrm{i}\,\mathrm{tr}\int_0^\sigma \mathrm{d}s\int\frac{\mathrm{d}^3k}{(2\pi)^3}G_F(k;s) \tag{9}$$

where the momentum-space Green function $G_F(k;s)$ has been used. One can now introduce the Riemann normal coordinate [22] with origin at any point in the spacetime. In this local coordinate system we use the weak curvature approximation for the Green function $G_F(k;s)$:

$$\begin{aligned} G_F(k;s) = {} & \frac{\gamma^a k_a + s}{k^2 - s^2} - \frac{R}{12}\frac{\gamma^a k_a + s}{(k^2-s^2)^2} + \frac{2}{3}R_{\mu\nu}k^\mu k^\nu\frac{(\gamma^a k_a + s)}{(k^2-s^2)^3} \\ & -\frac{1}{2}\gamma^a J^{cd}R_{cda\mu}k^\mu\frac{1}{(k^2-s^2)^2} \end{aligned} \tag{10}$$

where $J^{ab} = \frac{1}{4}[\gamma^a,\gamma^b]$, and the Latin and Greek indices refer to a local orthonormal frame and general coordinate system, respectively. Equation (10) is the linear approximation for the Green function $G_F(k;s)$ in the curvature R [11–13]. According to the well known method developed in [23], one should neglect any terms involving derivatives higher than those of the second order in the metric tensor expansion to obtain equation (10).

Let us now consider the effect of nonzero chemical potential μ on the system. It is common knowledge that the fermion-number density is directly related to the chemical potential μ. Mathematically, the presence of a nonzero chemical potential is realized by shifting the energy levels k_0 in the propagator $G_F(k;s)$ by the amount of μ [24]. Thus, we are able to evaluate the effective potential $U(\sigma)$ in equation (9) under the effects of both R and μ. Using the contour integration method [24], we can perform the integration over momentum k^μ. Denote I_1 as the integral of the first term in $G_F(k;s)$ over k and s. Its

calculation proceeds as follows. First, the procedure of integration over k_0, denoted as I_1', gives the result:

$$\begin{aligned} I_1'(k,s) &\equiv \mathrm{tr}\int \frac{\mathrm{d}k_0}{2\pi}\frac{\gamma^0(k_0+\mu)+\gamma^i k_i + s}{(k_0+\mu)^2 - E_k^2} \\ &= \frac{2}{\pi}\int_{-i\infty}^{i\infty}\frac{s\,\mathrm{d}z}{z^2-E_k^2} + \frac{2}{\pi}\oint_C \frac{s\,\mathrm{d}z}{z^2-E_k^2} \\ &= \frac{2}{\pi}\int_{-i\infty}^{i\infty}\frac{s\,\mathrm{d}z}{z^2-E_k^2} + \frac{2\mathrm{i}s}{E_k}\theta(\mu-E_k). \end{aligned} \tag{11}$$

Here, $E_k^2 \equiv k^2 + s^2$, the contour C is given in figure 1, and the unit step function $\theta(x) = 1$ for $x > 0, \theta(x) = 0$ for $x < 0$ has been used. Thus, we find

$$\begin{aligned} I_1 &\equiv -\mathrm{i}\int_0^\sigma \mathrm{d}s \int \frac{\mathrm{d}^2k}{(2\pi)^2} I_1'(k,s) \\ &= \sigma^2\left[\frac{\sigma}{3\pi} - \frac{\Lambda}{\pi^2}\right] + \theta(\mu-\sigma)\left[\frac{\mu}{2\pi}\sigma^2 - \frac{1}{3\pi}\sigma^3\right] + \theta(\sigma-\mu)\frac{\mu^3}{6} \end{aligned} \tag{12}$$

where Λ is the cut-off parameter. Here and in what follows, we can confine ourselves to the $\sigma \geqslant 0$ region owing to a reflection symmetry $\sigma \leftrightarrow -\sigma$ of the effective potential $U(\sigma)$. However, note that this symmetry is broken when the system selects one of the two ground states. In a similar way one finds the contributions I_2, I_3, I_4 of the remaining terms of $G_F(k;s)$ to the potential (9):

$$\begin{aligned} I_2 &= \frac{R}{24\pi}\left[-\sigma + \theta(\mu-\sigma)\left(\sigma - \frac{1}{2\mu}\sigma^2\right) + \theta(\sigma-\mu)\frac{\mu}{2}\right] \\ I_3 &= \frac{1}{12\pi}\left[R\sigma - R_{00}\left[\theta(\mu-\sigma)\left(\sigma - \frac{1}{2\mu}\sigma^2\right) + \theta(\sigma-\mu)\frac{\mu}{2}\right]\right] \\ &= \frac{1}{12\pi}R\sigma \\ I_4 &= 0. \end{aligned} \tag{13}$$

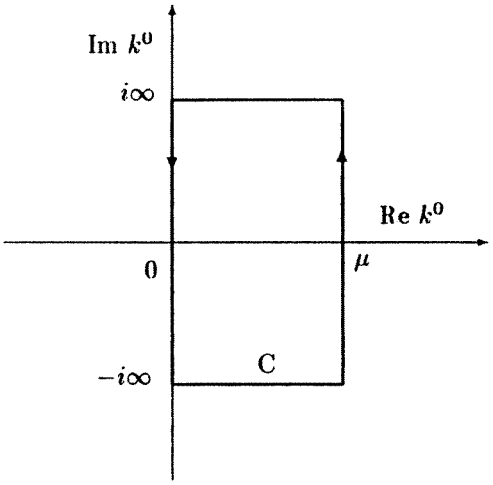

Figure 1. The contour C in the complex k^0 plane.

We confine ourselves to the $R_{00} = 0$ case without losing the generality of our discussion. Thus, in the third line of equation (13), we have set $R_{00} = 0$. However, the fourth line of equation (13) is due to a relation $\mathrm{tr}[\gamma^i \gamma^j \gamma^k] = 0$.

At this stage it is convenient to introduce the mass parameter M instead of the coupling constant λ in the following way [2]

$$\frac{1}{\lambda^2} \equiv 4 \int^{\Lambda} \frac{\mathrm{d}^3 k_E}{(2\pi)^3} \frac{1}{k_E^2 + M^2} = \frac{2}{\pi^2}\Lambda - \frac{1}{\pi} M. \tag{14}$$

Thus, we shall consider the case $\lambda > \lambda_c$ only, where $\lambda_c^{-2} \equiv 4 \int^{\Lambda} \mathrm{d}^3 k_E (2\pi)^{-3} k_E^{-2}$. Summing up all of the terms I_i in equations (12) and (13) and inserting the above equation into equation (9), one sees that the two Λ-dependent terms cancel out, and thus the finite effective potential to one-loop order is obtained. Then, the μ- and R-dependent one-loop contributions $U^1_{R\mu}(\sigma)$ to the potential $U(\sigma)$ are completely separated from the Minkowski-space result:

$$U(\sigma) = U_F(\sigma) + U^1_{R\mu}(\sigma) \tag{15}$$

where $U_F(\sigma)$ is the effective potential of the original theory in the flat Minkowski spacetime. Here

$$\begin{aligned} U_F(\sigma) &= \frac{\sigma^2}{3\pi}\left[\sigma - \frac{3}{2}M\right] \\ U^1_{R\mu}(\sigma) &= \frac{R}{24\pi}\sigma + \frac{1}{\pi}\theta(\mu - \sigma)\left[\left(\mu - \frac{2}{3}\sigma\right)\frac{\sigma^2}{2} + \frac{R}{24}\left(\sigma - \frac{\sigma^2}{2\mu}\right)\right] \\ &\quad + \frac{1}{6\pi}\theta(\sigma - \mu)\left[\mu\left(\mu^2 + \frac{R}{8}\right)\right]. \end{aligned} \tag{16}$$

In this expression one can find the following two facts. First, $U^1_{R\mu}(\sigma)$ is finite and, as $R, \mu \to 0$, $U^1_{R\mu}(\sigma)$ vanishes. Thus, the renormalization procedure is identical to the case of Minkowski spacetime. Second, in the limit $\mu, R \to 0$, $U(\sigma)$ is reduced to the Minkowski-space effective potential $U_F(\sigma)$.

It is well established that there are two distinct phases in the three-dimensional GN model [1, 2, 7]. For a weak coupling phase with the coupling $\lambda < \lambda_c$, we have $\langle\sigma\rangle = 0$. Thus, the fermions are massless and the chiral symmetry remains intact. However, for the strong coupling phase $\lambda > \lambda_c$, the σ field has a nonzero vacuum expectation value $\langle\sigma\rangle = M$, so the chiral symmetry of the expression (1) is dynamically broken and fermions acquire the mass, which is equal to the mass parameter M from equation (14).

For simplicity of our analysis in the following sections, we shall introduce the following rescaled dimensionless quantities defined as $\tilde{U}(x) \equiv \pi U(\sigma)/\mu^3$, $\tilde{R} \equiv R/\mu^2$, $x \equiv \sigma/\mu$, and $\bar{\mu} \equiv \mu/M$. In terms of these quantities, equation (15) is rewritten in the much simpler form:

$$\tilde{U}(x) = \begin{cases} \left(1 - \dfrac{1}{\bar{\mu}} - \dfrac{\tilde{R}}{24}\right)\dfrac{x^2}{2} + \dfrac{\tilde{R}x}{12} & \text{for } x < 1 \\ \left(x - \dfrac{3}{2\bar{\mu}}\right)\dfrac{x^2}{3} + \dfrac{\tilde{R}x}{24} + \dfrac{1}{6}\left(1 + \dfrac{\tilde{R}}{8}\right) & \text{for } x \geqslant 1 \end{cases} \tag{17}$$

where $\tilde{U}(x)$ is a continuous function at $x = 1$. We also wish to find the induced fermion mass $\langle\sigma\rangle$ as a function of curvature R and chemical potential μ. Then, the gap equation for

the fermion mass can be obtained by taking the derivative of the effective potential $\tilde{U}(x)$ with respect to x, and so we obtain

$$0 = \begin{cases} \left(1 - \dfrac{1}{\bar{\mu}} - \dfrac{\tilde{R}}{24}\right) x + \dfrac{\tilde{R}}{12} & \text{for } x < 1 \\ x^2 - \dfrac{x}{\bar{\mu}} + \dfrac{\tilde{R}}{24} & \text{for } x \geqslant 1. \end{cases} \tag{18}$$

3. Restoration of chiral symmetry

Now we shall analyse in detail the effective potential of equation (17) in order to investigate the phase structure of the model in the (R, μ) plane. The fermion mass $\langle\sigma\rangle$ shall be derived, which depends on R and μ, and the nature of the phase transitions shall be discussed. For clarity, we consider three distinct cases: $\mu \neq 0$ and $R = 0$, then $R \neq 0$ and $\mu = 0$, and finally $R \neq 0$ and $\mu \neq 0$.

3.1. The case $\mu \neq 0$ and $R = 0$

First let us examine the effect of nonzero chemical potential on the system. In the limit $R \to 0$, the effective potential of equation (17) is reduced to a simple form:

$$\tilde{U}(x) = \begin{cases} \left(1 - \dfrac{1}{\bar{\mu}}\right) \dfrac{x^2}{2} & \text{for } x < 1 \\ \left(x - \dfrac{3}{2\bar{\mu}}\right) \dfrac{x^2}{3} + \dfrac{1}{6} & \text{for } x \geqslant 1. \end{cases} \tag{19}$$

To see a phase transition as $\bar{\mu}$ increases from a broken phase to a symmetric one, it is necessary to examine the behaviour of $\tilde{U}(x)$ as a function of $\bar{\mu}$. It is possible to find the following two properties of $\tilde{U}(x)$. For $\bar{\mu} > 1$, $\tilde{U}(x)$ is a monotonically increasing function of x, and so the global minimum of $\tilde{U}(x)$ occurs at $x = 0$. While for $\bar{\mu} < 1$, $\tilde{U}(x)$ has a global minimum at $x = 1/\bar{\mu}$ with the value:

$$\tilde{U}\left(x = \frac{1}{\bar{\mu}}\right) = -\frac{1}{6}\frac{(1 - \bar{\mu}^3)}{\bar{\mu}^3}. \tag{20}$$

These facts indicate that the system undergoes a phase transition from the $\langle\sigma\rangle = M$ state to the $\langle\sigma\rangle = 0$ state at the critical value μ_c of the chemical potential, given as

$$\mu_c = M. \tag{21}$$

By solving the gap equation for the induced fermion mass, equation (18) with $R = 0$, one can find that

$$\langle\sigma\rangle = M \tag{22}$$

below μ_c, and $\langle\sigma\rangle = 0$ above μ_c. Except when $\mu = \mu_c$, the order parameter $\langle\sigma\rangle$ does not depend on the value of μ. That is, the value of order parameter $\langle\sigma\rangle$, which minimizes the potential, jumps discontinuously from $\langle\sigma\rangle = M$ to $\langle\sigma\rangle = 0$ at the transition point μ_c. Hence, at the point $\mu = \mu_c$ we have a first-order phase transition from a massive chirally broken phase to a massless chirally invariant phase of the model.

3.2. The case $R \neq 0$ and $\mu = 0$

In this case only the effect of curvature on the system shall be considered. In the limit $\mu \to 0$, the general effective potential of equation (17) has the following form

$$\begin{aligned} U(\sigma) &= U_F(\sigma) + U_R^1(\sigma) \\ &= \frac{\sigma^2}{3\pi}\left(\sigma - \frac{3}{2}M\right) + \frac{R}{24\pi}\sigma. \end{aligned} \tag{23}$$

This expression coincides with that obtained in [14, 18]. From equation (23) one can see that in the region of small values of σ the dominant contribution to $U(\sigma)$ comes from the R-dependent linear term in σ. Thus, for all values of $R > 0$ there is a potential barrier between $\sigma = 0$ and the second local minimum of $U(\sigma)$. As a result, it turns out that with the increase of curvature R the discontinuous phase transition occurs from a chirally broken phase to a symmetric one. The critical value of the curvature R_c, at which a first-order phase transition occurs, is defined by the following two conditions

$$U'(\sigma_0) = 0 \qquad \text{and} \qquad U(\sigma_0) = 0 \tag{24}$$

where σ_0 denotes the second nonzero local minimum of the potential. Furthermore, one can find that only for $R > R_c$ the minimum of the potential at the symmetric point $\sigma = 0$ is lower than the asymmetric local minimum at a nonzero σ_0. From the gap equation (18) with $\mu = 0$, one can evaluate the local minimum of the potential σ_0,

$$\sigma_0 = \frac{M}{2}\left(1 + \sqrt{1 - \frac{1}{6}\frac{R}{M^2}}\right) \tag{25}$$

which at the same time is equal to the fermion mass $\langle\sigma\rangle$, induced under the influence of curvature R for $R < R_c$ only. Thus, by applying the critical condition equation (24) to the effective potential of equation (23), one can obtain the critical curvature

$$R_c = 4.5M^2. \tag{26}$$

The phase transition under the influence of R is a first-order one since it occurs discontinuously. The same result for the R_c in the $(2+1)$-dimensional GN model in an arbitrary weakly curved spacetime was obtained in [11, 18].

3.3. The case $R \neq 0$ and $\mu \neq 0$

In this section we shall explore a general case wherein the system is specified by the curvature and finite chemical potential. To investigate the vacuum structure of the system when R and μ are varied, one must first examine the behaviour of the potential $\tilde{U}(x)$ as a function of R and μ. It is very helpful to sketch qualitatively the effective potential $\tilde{U}(x)$ from equation (17). For $\bar{\mu} > 1 (\mu > M)$ the global minimum of $\tilde{U}(x)$ only occurs at $x = 0$. While for $\bar{\mu} < 1$ $(\mu < M)$, the second local minimum of $\tilde{U}(x)$ certainly lies at a nonzero point. Therefore, when $\bar{\mu} < 1$, it turns out that the system undergoes a phase transition from the $\langle\sigma\rangle \neq 0$ vacuum state to the $\langle\sigma\rangle = 0$ state at a certain critical curvature $\tilde{R}_c$ depending on μ. Using a more detailed analysis of the effective potential $\tilde{U}(x)$ in equation (17), one can see that until the system approaches the critical point with the increase of curvature, the second local minimum of $\tilde{U}(x)$ occurs only in the region $x > 1$. Therefore, in the procedure of determining the critical value of the curvature $\tilde{R}_c$, the effective potential needs to be considered only in the $x > 1$ region in equation (17).

In this case we can obtain the critical curvature R_c also using the condition given in equation (24), with the only change $\sigma_0 \to x_0$, where x_0 denotes the second local minimum

of the potential. That is, in this case the phase transition under investigation is also a first-order one. As can easily be checked from the gap equation (18), the second minimum lies at the point

$$x_0 = \frac{1}{2\bar{\mu}}\left(1 + \sqrt{1 - \frac{\tilde{R}\bar{\mu}^2}{6}}\right). \tag{27}$$

Thus, the critical condition of equation (24) with this value for x_0 leads to the self-consistent relation on the critical curvature $\tilde{R}_c$:

$$16x_0^3 - \frac{24x_0^2}{\bar{\mu}} + (2x_0 + 1)\tilde{R}_c + 8 = 0 \tag{28}$$

where x_0 has the value given in equation (27), with $\tilde{R}$ replaced by $\tilde{R}_c$. The numerical solutions of equation (28) are illustrated in figure 2. Note, that with $\mu \to 0$, the R_c approaches $4.5M^2$ and with $R \to 0$, the μ_c approaches M. These limiting cases have already been discussed in the previous sections 3.1 and 3.2. Equation (27) suggests that the induced fermion mass $\langle\sigma\rangle$, with $\langle\sigma\rangle = \mu x_0$, only depends on the curvature R. That is, $\langle\sigma\rangle$ does not depend on μ, and thus has the same expression as in equation (25). In figure 3, the effective potentials are given for four distinct values of R at fixed $\mu = \frac{M}{2}$.

4. Summary and discussion

In this paper we have derived the effective potential of the three-dimensional Gross–Neveu model in the curved spacetime of the form $R^1 \times M^2$ and with taking into account the

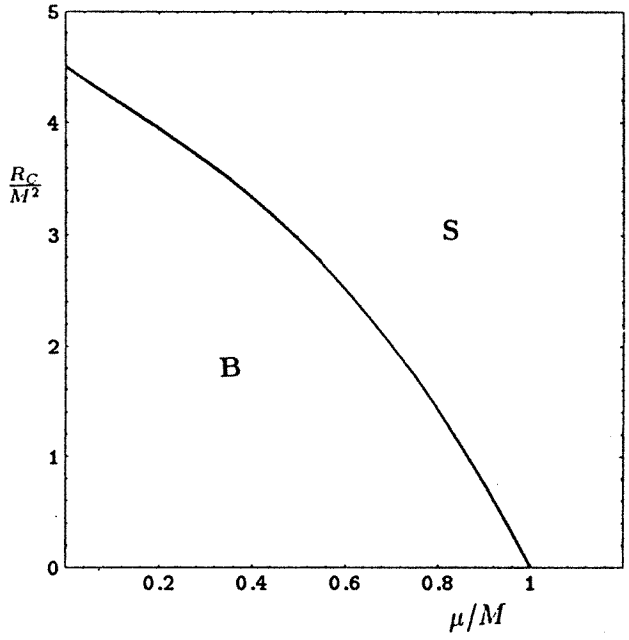

Figure 2. The critical curvature R_c/M^2 as a function of nonzero chemical potential μ/M. In region B, chiral symmetry is broken and fermions acquire dynamical masses, while in S, the symmetry is restored by the curvature effect, and the fermions become massless.

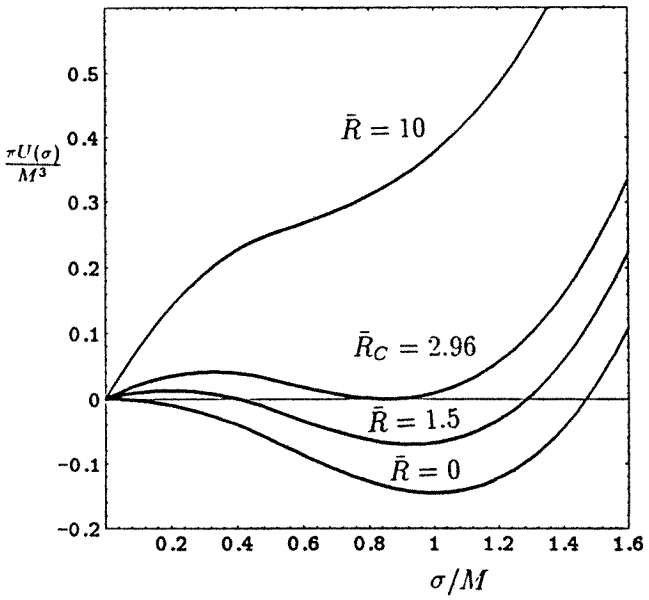

Figure 3. The effective potential $\pi U(\sigma)/M^3$ as a function of σ/M at the fixed value of $\mu/M = \frac{1}{2}$. Four interesting cases of $\bar{R}$, where $\bar{R} \equiv R/M^2$, are considered, and the critical curvature $\bar{R}_c$ is then obtained numerically: $\bar{R}_c = 2.96$.

chemical potential μ as well. Then, the critical curvature R_c at which dynamical symmetry breaking disappears, was determined in terms of the induced fermion mass M in the limit $R, \mu \to 0$ and at nonzero chemical potential μ, as given in figure 2.

In sections 3.1 and 3.3 it was shown that at fixed curvature $R < R_c$ a critical value of the chemical potential $\mu_c(R)$ was available. In this critical point the system undergoes a chiral phase transition of the first order. We also observed that the order parameter $\langle\sigma\rangle$ of the phase transition, corresponding to the minimum of the potential, did not depend on the value of μ, except at the critical value $\mu = \mu_c$, even though the phase transition was induced by the chemical potential. This phenomenon is connected to the fact that the composed field $\sigma \sim \bar{\psi}\psi$ is a real field and carries no charge. It was also observed in the two-dimensional GN model in $R^1 \times S^1$ spacetime [21], however, in that model there is another massive phase, in which fermion mass is a μ-dependent quantity.

Furthermore, by analysing equation (21) one can come across an interesting fact. Let us suppose that the particle density $N(\mu)$ of the system is not zero. In this case, by analogy with condensed matter physics, the chemical potential corresponds to the Fermi energy, that must be greater than the minimal energy of one fermion, i.e. $\mu > \langle\sigma\rangle$. Hence, at $\mu < \mu_c = M$ there is a massive phase of the theory ($\langle\sigma\rangle = M$), at which $N(\mu)$ equals zero. At $\mu > \mu_c = M$ the symmetric phase of the model is arranged. Here fermion density in the vacuum is not zero, and at the critical point $\mu = M$, the function $N(\mu)$ is a discontinuous one. Recently, a similar nonanalytic behaviour of the Chern–Simons coefficient in the presence of chemical potential has been found in a $(2 + 1)$-dimensional QED [25].

In this paper we have dealt with noncompact manifolds M^2 only. We wish to remark that for the case with compact surfaces M^2 the weak curvature technique is not applicable. Indeed, in [14] the three-dimensional GN model in $R^1 \times S^2$ spacetime and at nonzero

temperature was considered. There the exact expression for the effective potential was calculated and it was shown that chiral phase transition in $R^1 \times S^2$ spacetime is always the second-order one, even at sufficiently small R. At the same time in the framework of weak curvature expansion one can find the first-order chiral phase transition. Hence, as was pointed out in [18], global topology of low-dimensional spacetimes plays an essential role in critical phenomena. However, in four-dimensional spacetime the weak curvature approximation is a rather good one in cases with compact as well as noncompact spaces [18].

In section 3.2 we have shown that $R_c = 4.5M^2$ at $\mu = 0$. This can roughly be seen from the following two facts. First, on dimensional grounds the critical curvature R_c must be proportional to the square of some quantity with the dimension of mass. Second, the effective potential for the composite σ field in equation (23) has two parameters R and M, and so, the remaining parameter apart from R in this theory is M. Note, that our value for R_c is found in a weak curvature limit, and thus its more accurate value can be obtained by considering higher order corrections over the scalar curvature R. However, in such improved schemes, it is expected that the system still shows the same qualitative properties as those found in the previous sections, including the occurrence of a first-order phase transition. (In [18] some speculations about the validity of the weak curvature expansion for large values of R are presented.)

Finally, one may consider the case of negative curvature since this method has the advantage of being applicable to any metric. Then, equation (25) indicates that under the effect of negative curvature R the minimum of the potential is located further from the origin than without the curvature effect. Therefore, in this case the symmetry restoring phase transition does not happen.

We hope that the above results may be useful for condensed matter physics and for astrophysical applications, in particular for the description of different phenomena in the core of neutron stars.

Acknowledgments

We are grateful to Professor S D Odintsov and Dr P A Saponov for reading the manuscript and for making some critical remarks as well as to Professor V P Gusynin for useful comments. DKK thanks Professor K-S Soh, Professor C K Kim and Professor J H Yee for helpful discussions.

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